An elastoplastic framework for granular materials becoming cohesive through mechanical densification. Part II - the formulation of elastoplastic coupling at large strain

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Abstract

The two key phenomena occurring in the process of ceramic powder compaction are the progressive gain in cohesion and the increase of elastic stiffness, both related to the development of plastic deformation. The latter effect is an example of 'elastoplastic coupling', in which the plastic flow affects the elastic properties of the material, and has been so far considered only within the framework of small strain assumption (mainly to describe elastic degradation in rock-like materials), so that it remains completely unexplored for large strain. Therefore, a new finite strain generalization of elastoplastic coupling theory is given to describe the mechanical behaviour of materials evolving from a granular to a dense state.

The correct account of elastoplastic coupling and of the specific characteristics of materials evolving from a loose to a dense state (for instance, nonlinear –or linear– dependence of the elastic part of the deformation on the forming pressure in the granular –or dense– state) makes the use of existing large strain formulations awkward, if even possible. Therfore, first, we have resorted to a very general setting allowing general transformations between work-conjugate stress and strain measures; second, we have introduced the multiplicative decomposition of the deformation gradient and, third, employing isotropy and hyperelasticity of elastic response, we have obtained a relation between the Biot stress and its 'total' and 'plastic' work-conjugate strain measure. This is a key result, since it allows an

immediate achievement of the rate elastoplastic constitutive equations. Knowing the general form of these equations, all the specific laws governing the behaviour of ceramic powders are finally introduced as generalizations of the small strain counterparts given in Part I of this paper.

Keywords: Elastoplasticity; Large strains; Granular materials; Mechanical densification; Forming; Ceramic Materials.

1 Introduction

Mechanical cold compaction of ceramic powder involves the transition from a granular to a dense state. During this process, and strictly related to the development of permanent deformations, both the cohesion *and elastic stiffness* of the material increase. This occurs also under isostatic compression (which does not involve deviatoric strain) and is believed to be related at the microscale to the increase of the contact area between the grains (therefore, the effect should not be confused with a large strain effect). The increase of elastic stiffness with permanent deformation is a sort of 'inverse damage', which can be described making recourse to the concept of elastoplastic coupling, originarily invented to model elastic degradation, and employed in the Part I of this paper to describe the stiffening during plastic deformation of ceramic powder. However, a large strain formulation of elastoplastic coupling has never been attempted. That this formulation is not trivial can be deduced from the fact that elastic characteristics have been assumed to be independent of plastic deformation in all elastoplastic models proposed for soils (Borja and Tamagnini, 1998; Callari et al. 1998; Rouainia and Muir Wood, 2000; Ortiz and Pandolfi, 2004) and in more general contexts (see among others: Simo and Miehe, 1992; Peric et al. 1992; Schieck and Stumpf, 1993; Simo and Meschke, 1993; Ibrahimbegovic, 1994).

Since the existing large strain formulations do not appear to be easily generalizable to admit a coupling between elastic and plastic deformations, we have recurred to the early formulation by Hill and Rice (1973) (see also Hill, 1978; Petryk and Thermann 1985; Bigoni, 1996; 2000), which (although not explicitely mentioned) has been formulated in such a generality to include coupling. Following this approach, the level of generality is so high that the following choices are not required: stress/strain measures [except that these are work-conjugate (Hill, 1968)], elastic and plastic strain decomposition, elastic law, yield function, flow and hardening rules. After this framework is provided, the multiplicative strain decomposition of Lee (1969) and Willis (1969) is introduced. At this point, assuming that the elastic response be hyperelastic and isotropic we have proved that a general relation exists, in which the Biot stress is related to its work-conjugate 'total' and 'plastic' strain measure (and to a generic set of scalar hidden variables). This achievement turns out to be crucial since it allows immediate use of the general

formulation previously developed[1]. Finally, the coupling and hardening laws, the yield function and all constitutive relations provided in Part I of this paper for the infinitesimal theory are consistently generalized to include large strains.

2 The skeleton of large strain elastoplasticity

2.1 Some preliminaries on work conjugacy

A broad constitutive framework for isothermal and time independent large elastoplastic deformations is presented, based on the concept of work coniugacy in the Hill sense (1968, 1978). In particular, employing Ogden's (1984) notation, a pair of symmetric, Lagrangean, stress $\boldsymbol{T}^{(m)}$ and strain $\boldsymbol{E}^{(m)}$ measures[2] are work-conjugate when the stress power density per unit volume in the reference configuration can be expressed as

$$\boldsymbol{T}^{(m)} \cdot \dot{\boldsymbol{E}}^{(m)} = \boldsymbol{S} \cdot \dot{\boldsymbol{F}}, \tag{1}$$

where a dot over a symbol denotes material time derivative, $\boldsymbol{F}$ is the deformation gradient and $\boldsymbol{S}$ the first Piola-Kirchhoff stress tensor

$$\boldsymbol{S} = J\boldsymbol{T}\boldsymbol{F}^{-T} = \boldsymbol{K}\boldsymbol{F}^{-T}, \tag{2}$$

in which $J = \det\boldsymbol{F}$ and $\boldsymbol{T}$ and $\boldsymbol{K} = J\boldsymbol{T}$ are the Cauchy and Kirchhoff stresses, respectively.

For integer (positive, null or negative) exponent m, we introduce the following Lagrangean strain measures

$$\begin{cases} \boldsymbol{E}^{(m)} = \dfrac{1}{m}(\boldsymbol{U}^m - \boldsymbol{I}), & \text{if } \ m \neq 0, \\ \boldsymbol{E}^{(0)} = \log \boldsymbol{U}, & \text{if } \ m = 0, \end{cases} \tag{3}$$

where the logarithm of a tensor is defined as in (Ogden, 1984) and

$$\boldsymbol{U} = (\boldsymbol{F}^T\boldsymbol{F})^{1/2}, \tag{4}$$

is the right stretch tensor. For a given m, $\boldsymbol{E}^{(m)}$ is defined by (3) and the corresponding work-conjugate stress measure $\boldsymbol{T}^{(m)}$ can be defined imposing eqn. (1). For instance, for

[1] The law between Biot stress and its work-conjugate strain measure could obviously be transformed into different (work-conjugate) stress/strain measures, but this would be cumbersome and useless, since the generality of the Hill and Rice (1973) approach allow us to use the obtained law directly.

[2] The notation $\boldsymbol{T}^m = \underbrace{\boldsymbol{T}\boldsymbol{T}...\boldsymbol{T}}_{m \text{ times}}$ (or $\boldsymbol{E}^m$) should not be confused with $\boldsymbol{T}^{(m)}$ (or $\boldsymbol{E}^{(m)}$).

$m = 2$, the Green-Lagrange strain results from eqn. (3) and the eqn. (1) provides for $\boldsymbol{T}^{(2)}$ the second Piola-Kirchhoff stress tensors,

$$\boldsymbol{E}^{(2)} = \frac{1}{2}\left(\boldsymbol{U}^2 - \boldsymbol{I}\right), \quad \text{conjugate to} \quad \boldsymbol{T}^{(2)} = \boldsymbol{F}^{-1}\boldsymbol{K}\boldsymbol{F}^{-T}. \tag{5}$$

A conjugate pair of stress and strain that will become useful later is formed by the Biot stress tensor $\boldsymbol{T}^{(1)}$ and the strain measure $\boldsymbol{E}^{(1)}$, defined as

$$\boldsymbol{E}^{(1)} = \boldsymbol{U} - \boldsymbol{I}, \quad \text{conjugate to} \quad \boldsymbol{T}^{(1)} = \frac{1}{2}\left(\boldsymbol{T}^{(2)}\boldsymbol{U} + \boldsymbol{U}\boldsymbol{T}^{(2)}\right). \tag{6}$$

It is well-known however that it is not always the easy task of the two above examples to obtain the stress measure conjugated to a given strain of the form (3). For instance, the conjugate of the logarithmic strain $\boldsymbol{E}^{(0)}$ has a very complex form (Hoger, 1987), which simplifies to the rotated stress only when the two measures result coaxial, namely,

$$\boldsymbol{E}^{(0)} = \log \boldsymbol{U} \quad \text{conjugate to} \quad \boldsymbol{T}^{(0)} = \boldsymbol{R}^T\boldsymbol{K}\boldsymbol{R}, \tag{7}$$

(where $\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1}$ is the rotation tensor) if and only if the following coaxiality condition holds true

$$\boldsymbol{E}^{(0)}\boldsymbol{T}^{(0)} = \boldsymbol{T}^{(0)}\boldsymbol{E}^{(0)} \quad \Longleftrightarrow \quad (\log\boldsymbol{V})\boldsymbol{K} = \boldsymbol{K}\log\boldsymbol{V}, \tag{8}$$

where $\boldsymbol{V}$ is the left stretch tensor, so that $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}$ (note that the above equivalence is an immediate consequence of the fact that the logarithmic function is isotropic). Condition (8) is satisfied for isotropic elasticity, but may be not in more general contexts, such as for instance elastoplasticity (Sansour, 2001). It may be instructive for subsequent considerations to note that, when the coaxiality condition (8) holds true, the following relation

$$\boldsymbol{T}^{(0)} \cdot \dot{\boldsymbol{E}}^{(0)} = \boldsymbol{K} \cdot (\log\boldsymbol{V})^{\cdot} \tag{9}$$

can be proved (Ogden, 1982) showing that the Eulerian stress and strain measures $\boldsymbol{K}$ and $\log\boldsymbol{V}$ are work-conjugate (Hill, 1968).

2.2 The basic assumptions of elastoplasticity

Following Bigoni (2000), inelastic materials are considered that may at any stage of deformation exhibit a purely elastic response for appropriate loading. For these materials, elastic response is assumed to be a one-to-one relation between a work-conjugate pair $\boldsymbol{T}^{(m)}$ and $\boldsymbol{E}^{(m)}$

$$\boldsymbol{T}^{(m)} = \hat{\boldsymbol{T}}^{(m)}(\boldsymbol{E}^{(m)}, \mathcal{K}), \quad \boldsymbol{E}^{(m)} = \hat{\boldsymbol{E}}^{(m)}(\boldsymbol{T}^{(m)}, \mathcal{K}), \tag{10}$$

where $\hat{\boldsymbol{T}}^{(m)}$ and $\hat{\boldsymbol{E}}^{(m)}$ are functionals depending on the prior history of inelastic deformation through the unspecified set $\mathcal{K}$ of variables of generic tensorial nature (thus embracing

scalars, vectors, second-order tensors and possibly higher-order tensors). For a purely elastic deformation rate, $\mathcal{K}$ remains fixed, so that we have

$$\dot{\boldsymbol{T}}^{(m)} = \mathbb{E}[\dot{\boldsymbol{E}}^{(m)}], \quad \dot{\boldsymbol{E}}^{(m)} = \mathbb{M}[\dot{\boldsymbol{T}}^{(m)}], \tag{11}$$

where the fourth-order tensors $\mathbb{E}$ and $\mathbb{M}$ possess the minor symmetries induced by $\boldsymbol{E}$ and $\boldsymbol{T}$, while the major symmetry is not a-priori requested (differently from Hill and Rice, 1973). They are defined as

$$\mathbb{E}(\boldsymbol{E}^{(m)}, \mathcal{K}) = \frac{\partial \hat{\boldsymbol{T}}^{(m)}}{\partial \boldsymbol{E}^{(m)}}, \quad \mathbb{M}(\boldsymbol{T}^{(m)}, \mathcal{K}) = \frac{\partial \hat{\boldsymbol{E}}^{(m)}}{\partial \boldsymbol{T}^{(m)}}. \tag{12}$$

Tensors (12) obviously satisfy

$$\mathbb{E} = \mathbb{M}^{-1}. \tag{13}$$

For an increment involving elastic and inelastic strain rates, we may write

$$\dot{\boldsymbol{T}}^{(m)} = \mathbb{E}[\dot{\boldsymbol{E}}^{(m)}] - \dot{\Lambda}\mathbb{E}[\boldsymbol{P}], \quad \dot{\boldsymbol{E}}^{(m)} = \mathbb{M}[\dot{\boldsymbol{T}}^{(m)}] + \dot{\Lambda}\boldsymbol{P}, \tag{14}$$

where $\boldsymbol{P} \in \mathsf{Sym}$ sets the 'direction' (or the 'mode') of the irreversible deformation, which is given by

$$\dot{\Lambda}\boldsymbol{P} = -\mathbb{E}^{-1}\frac{\partial \hat{\boldsymbol{T}}^{(m)}}{\partial \mathcal{K}}[\dot{\mathcal{K}}] = \frac{\partial \hat{\boldsymbol{E}}^{(m)}}{\partial \mathcal{K}}[\dot{\mathcal{K}}]. \tag{15}$$

The scalar $\dot{\Lambda} \geq 0$ appearing in eqn. (14) is the plastic multiplier and vanishes for purely elastic response, namely, when $\dot{\mathcal{K}} = 0$.

A yield surface is assumed at each $\mathcal{K}$, which may be alternatively expressed in stress and strain spaces as

$$f_{\boldsymbol{T}^{(m)}}(\boldsymbol{T}^{(m)}, \mathcal{K}) \leq 0 \ \text{ or as } \ f_{\boldsymbol{E}^{(m)}}(\boldsymbol{E}^{(m)}, \mathcal{K}) \leq 0, \tag{16}$$

thus defining regions of the $\boldsymbol{T}^{(m)}$ or $\boldsymbol{E}^{(m)}$ space, respectively, within which the response is elastic.

2.3 Direct rate constitutive equations

Prager's consistency condition requires

$$\dot{f}_{\boldsymbol{T}^{(m)}} = \dot{f}_{\boldsymbol{E}^{(m)}} = 0, \tag{17}$$

when inelastic strain rate is different from zero. As a consequence, employing the stress space representation, the elastoplastic incremental constitutive equations can be written as

$$\dot{\boldsymbol{T}}^{(m)} = \begin{cases} \mathbb{E}[\dot{\boldsymbol{E}}^{(m)}] - \dfrac{1}{g} < \boldsymbol{Q} \cdot \mathbb{E}[\dot{\boldsymbol{E}}^{(m)}] > \mathbb{E}[\boldsymbol{P}] & \text{if } f_{\boldsymbol{T}^{(m)}}(\boldsymbol{T}^{(m)}, \mathcal{K}) = 0, \\ \mathbb{E}[\dot{\boldsymbol{E}}^{(m)}] & \text{if } f_{\boldsymbol{T}^{(m)}}(\boldsymbol{T}^{(m)}, \mathcal{K}) < 0, \end{cases} \tag{18}$$

where the operator $< \cdot >$ denotes the Macaulay brackets, i.e. $\forall\ \alpha \in \mathbf{R}$, $< \alpha >= (\alpha + |\alpha|)/2$. Moreover, the symmetric second-order tensor

$$\boldsymbol{Q} = \frac{\partial f_{\boldsymbol{T}^{(m)}}}{\partial \boldsymbol{T}^{(m)}}, \tag{19}$$

is the yield function gradient and the plastic modulus

$$g = h + \boldsymbol{Q} \cdot \mathbb{E}[\boldsymbol{P}], \tag{20}$$

is assumed to be strictly positive (a negative plastic modulus would correspond to a so-called locking material, not considered here). In the Hill (1967) notation, the hardening modulus h in (20) describes:

- *hardening* when positive,
- *softening* when negative,
- *perfect plasticity* when null.

The hardening modulus is defined as

$$\dot{\Lambda} h = -\frac{\partial f_{\boldsymbol{T}^{(m)}}}{\partial \mathcal{K}} \cdot \dot{\mathcal{K}}, \tag{21}$$

and, as Hill (1967) remarks, hardening and softening are not measure-invariant concepts, in the sense that h depends on the choice of $\boldsymbol{T}^{(m)}$ and $\boldsymbol{E}^{(m)}$. Therefore, the above nomenclature is, to some extent, arbitrary. Moreover, we remark that, in addition to h, also $\boldsymbol{Q}$, $\boldsymbol{P}$ and $\mathbb{E}$ are measure-dependent. On the contrary, the plastic modulus g can be shown to be measure-independent (Hill, 1967; Petryk, 2000). Note also that all quantities appearing in the rate equations (18) fully depend on the entire path of deformation reckoned from some ground state.

2.4 Inverse rate constitutive equations

Under the assumption of positive hardening, the rate equations (18) can be inverted to relate the material derivative of the strain measure to the material derivative of the work-conjugate stress. In particular, taking the scalar product of the first equation in (18) with $\boldsymbol{Q}$ gives

$$\boldsymbol{Q} \cdot \dot{\boldsymbol{T}}^{(m)} = \boldsymbol{Q} \cdot \mathbb{E}[\dot{\boldsymbol{E}}^{(m)}] - \frac{\boldsymbol{Q} \cdot \mathbb{E}[\boldsymbol{P}]}{g} < \boldsymbol{Q} \cdot \mathbb{E}[\dot{\boldsymbol{E}}^{(m)}] > . \tag{22}$$

In the case when $h > 0$, we note that

$$\operatorname{sgn}(\boldsymbol{Q} \cdot \mathbb{E}[\dot{\boldsymbol{E}}^{(m)}]) = \operatorname{sgn}(\boldsymbol{Q} \cdot \dot{\boldsymbol{T}}^{(m)}).$$

Therefore, assuming positive hardening, $h > 0$, and using (22), we obtain the inverse constitutive equations

$$\dot{\boldsymbol{E}}^{(m)} = \begin{cases} \mathbb{M}[\dot{\boldsymbol{T}}^{(m)}] + \dfrac{1}{h} < \boldsymbol{Q} \cdot \dot{\boldsymbol{T}}^{(m)} > \boldsymbol{P} & \text{if } \ f_{\boldsymbol{T}^{(m)}}(\boldsymbol{T}^{(m)}, \mathcal{K}) = 0, \\ \mathbb{M}[\dot{\boldsymbol{T}}^{(m)}] & \text{if } \ f_{\boldsymbol{T}^{(m)}}(\boldsymbol{T}^{(m)}, \mathcal{K}) < 0, \end{cases} \tag{23}$$

The rate constitutive equations (18) or (23) represent a broad constitutive framework, within which

> all possible choices of $\boldsymbol{T}^{(m)}$ and $\boldsymbol{E}^{(m)}$ are equivalent and the requirement of material frame indifference (Truesdell and Noll, 1965) is never violated.

Moreover, the framework is so general that it does not imply any particular choice of

- elastic and plastic strain decomposition,
- hypo- or hyper- elastic law,
- yield function, flow and hardening rules.

It is however clear that in order to set up the constitutive modelling of a particular material, we need to introduce specific laws. This objective will be pursued in three steps of deceasing generality in the following: first, we will introduce the multiplicative elastic and plastic strain decomposition and requirement of objectivity and isotropy of the elastic constitutive law; second, a form of elastic constitutive equation will be proposed, depending on plastic deformation and thus capable of describing the elastic behaviour of granular and dense materials; third, yield function and hardening laws are introduced as simple generalizations of the rules formulated in Part I of this paper under the small strain assumption.

3 The multiplicative decomposition and the elastic law

The multiplicative decomposition of deformation gradient $\boldsymbol{F}$ into elastic $\boldsymbol{F}_e$ and plastic $\boldsymbol{F}_p$ components introduced by Lee (1969) and Willis (1969) is adopted (Fig. 1).

$$\boldsymbol{F} = \boldsymbol{F}_e \boldsymbol{F}_p. \tag{24}$$

According to eqn. (24), using the left polar decomposition $\boldsymbol{F} = \boldsymbol{V}\boldsymbol{R}$, we introduce the elastic and plastic left stretch and rotation tensors $\boldsymbol{V}_e$, $\boldsymbol{V}_p$, $\boldsymbol{R}_e$ and $\boldsymbol{R}_p$, satisfying

$$\boldsymbol{F}_e = \boldsymbol{V}_e \boldsymbol{R}_e, \quad \boldsymbol{F}_p = \boldsymbol{V}_p \boldsymbol{R}_p, \tag{25}$$

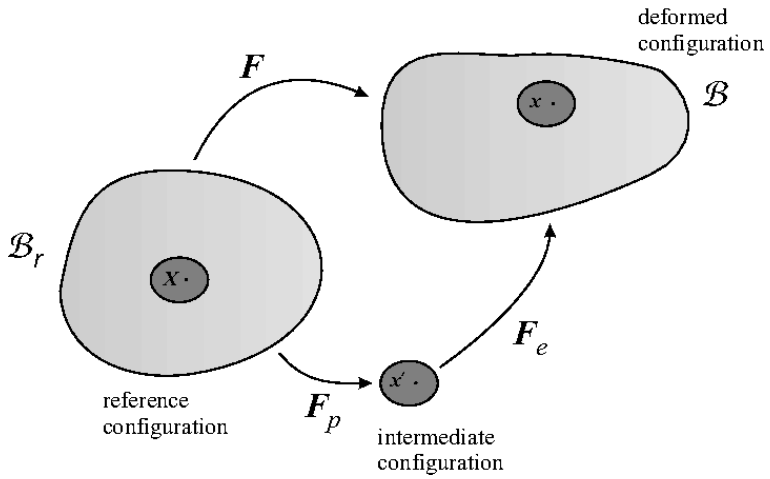

Figure 1: Reference, current and intermediate configurations.

while using the right polar decomposition $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$, the right elastic and plastic stretch tensors $\boldsymbol{U}_e$ and $\boldsymbol{U}_p$ are defined so that they satisfy

$$\boldsymbol{F}_e = \boldsymbol{R}_e\boldsymbol{U}_e, \quad \boldsymbol{F}_p = \boldsymbol{R}_p\boldsymbol{U}_p. \tag{26}$$

A crucial expedient, employed also by Ortiz and Pandolfi (2004), to describe the behaviour of granular materials is to refer to the logarithmic strains defined as

$$\boldsymbol{\epsilon} = \log \boldsymbol{V}, \quad \boldsymbol{\epsilon}_e = \log \boldsymbol{V}_e, \quad \boldsymbol{\epsilon}_p = \log \boldsymbol{V}_p, \tag{27}$$

and

$$\boldsymbol{E}^{(0)} = \log \boldsymbol{U}, \quad \boldsymbol{E}^{(0)}_e = \log \boldsymbol{U}_e, \quad \boldsymbol{E}^{(0)}_p = \log \boldsymbol{U}_p. \tag{28}$$

The interest in employing definitions (27) and (28) is that these allow a decoupling between the volumetric logarithmic elastic and plastic deformations, namely

$$\operatorname{tr} \boldsymbol{\epsilon} = \operatorname{tr} \boldsymbol{\epsilon}_e + \operatorname{tr} \boldsymbol{\epsilon}_p = \operatorname{tr} \boldsymbol{E}^{(0)} = \operatorname{tr} \boldsymbol{E}^{(0)}_e + \operatorname{tr} \boldsymbol{E}^{(0)}_p, \tag{29}$$

which, employing the usual definition of J and noting the property (for every symmetric tensor $\boldsymbol{A}$)

$$\operatorname{tr}(\log \boldsymbol{A}) = \log(\det \boldsymbol{A}), \tag{30}$$

can be written as

$$\log J = \log J_e + \log J_p. \tag{31}$$

4 Objectivity and isotropy of elastic response

We refer now to an isotropic elastic law relating the Kirchhoff stress $\boldsymbol{K}$ to the elastic deformation gradient $\boldsymbol{F}_e$ in the generic form

$$\boldsymbol{K} = \hat{\boldsymbol{K}}(\boldsymbol{F}_e, k_i), \tag{32}$$

where *function $\hat{\boldsymbol{K}}$ may depend also on generic plastic scalar variables k_i, assumed invariant with respect to every symmetry group of the material and change in observer.*

In general, the elastic response must be objective, but in addition we assume for simplicity that the elastic response be isotropic. The latter requirement implies the coaxiality condition (8), a requisite more important than it may appear, since it ensures the work coniugacy (7) and its Eulerian counterpart. Therefore, function $\hat{\boldsymbol{K}}$ is assumed to satisfy:

P1. The objectivity requirement

$$\hat{\boldsymbol{K}}(\boldsymbol{F}_e, k_i) = \boldsymbol{R}^T \hat{\boldsymbol{K}}(\boldsymbol{R}\boldsymbol{F}_e, k_i)\boldsymbol{R}, \quad \forall \boldsymbol{R} \in \mathsf{Orth}^+. \tag{33}$$

P2. The isotropy requirement

$$\hat{\boldsymbol{K}}(\boldsymbol{F}_e, k_i) = \hat{\boldsymbol{K}}(\boldsymbol{F}_e\boldsymbol{R}, k_i), \quad \forall \boldsymbol{R} \in \mathsf{Orth}^+. \tag{34}$$

Though the generalization of the formulation to anisotropy of the elastic response is definitely important to capture certain experimental evidence connected to the development of various form of instabilities [see Gajo et al. (2004) for a discussion relative to the infinitesimal theory in the context of granular media], we note that this generalization is for the moment lacking for granular material subject to large strains, even in the relatively simple setting in which cohesion and coupling are neglected.

As a consequence of isotropy, eqn. (34), the rotation in the left polar decomposition does not alter the values of function $\hat{\boldsymbol{K}}$,

$$\hat{\boldsymbol{K}}(\boldsymbol{F}_e, k_i) = \hat{\boldsymbol{K}}(\boldsymbol{V}_e, k_i), \tag{35}$$

so that function $\hat{\boldsymbol{K}}$ depends only on the elastic left stretch tensor. Noting the identity

$$\boldsymbol{F}_e = \boldsymbol{R}\boldsymbol{U}\boldsymbol{U}_p^{-1}\boldsymbol{R}_p^T, \tag{36}$$

isotropy and objectivity allow us to introduce the following transformations

$$\boldsymbol{K} = \hat{\boldsymbol{K}}(\boldsymbol{R}\boldsymbol{U}\boldsymbol{U}_p^{-1}\boldsymbol{R}_p^T, k_i) = \hat{\boldsymbol{K}}(\boldsymbol{R}\boldsymbol{U}\boldsymbol{U}_p^{-1}, k_i) = \boldsymbol{R}\hat{\boldsymbol{K}}(\boldsymbol{U}\boldsymbol{U}_p^{-1}, k_i)\boldsymbol{R}^T, \tag{37}$$

so that we get

$$\boldsymbol{K} = \hat{\boldsymbol{K}}(\boldsymbol{F}_e, k_i) = \boldsymbol{R}\hat{\boldsymbol{K}}(\boldsymbol{U}\boldsymbol{U}_p^{-1}, k_i)\boldsymbol{R}^T. \tag{38}$$

Employing now the rotated stress $\boldsymbol{R}^T\boldsymbol{K}\boldsymbol{R}$, the constitutive law (32) can be written in the form

$$\boldsymbol{R}^T\boldsymbol{K}\boldsymbol{R} = \hat{\boldsymbol{K}}(\boldsymbol{U}\boldsymbol{U}_p^{-1}, k_i), \tag{39}$$

relating the rotated stress to the total and plastic right stretch tensors.

Now, the rotated stress is related to the Biot stress through (Ogden, 1984)

$$\boldsymbol{T}^{(1)} = \frac{1}{2}\left(\boldsymbol{U}^{-1}\boldsymbol{R}^T\boldsymbol{K}\boldsymbol{R} + \boldsymbol{R}^T\boldsymbol{K}\boldsymbol{R}\boldsymbol{U}^{-1}\right), \tag{40}$$

so that in conclusion we obtain the elastic constitutive law in the form

$$\boldsymbol{T}^{(1)} = \frac{1}{2}\left(\boldsymbol{U}^{-1}\hat{\boldsymbol{K}}(\boldsymbol{U}\boldsymbol{U}_p^{-1}, k_i) + \hat{\boldsymbol{K}}(\boldsymbol{U}\boldsymbol{U}_p^{-1}, k_i)\boldsymbol{U}^{-1}\right), \tag{41}$$

relating the Biot stress to the global and plastic right stretch tensors.

Since eqn. (6) shows that tensor $\boldsymbol{E}^{(1)}$ is the right stretch tensor $\boldsymbol{U}$ with the identity subtracted,

> eqn. (41) expresses a relation between the two work-conjugate measures $\boldsymbol{T}^{(1)}$ and $\boldsymbol{E}^{(1)}$ of the type (10), in which the set $\mathcal{K}$ is now including $\boldsymbol{U}_p^{-1}$ and the scalars k_i.

5 Formulation of the rate model

Until this point, all the equations are at a high level of generality; now, to develop the model for powder densification, further specific laws are introduced, including the particular hyperelastic-plastic coupling rule, yield function and hardening laws. Since these laws are essentially extensions of those already employed in the small strain formulation, details on physical motivations determining the specific choices will be omitted for conciseness.

5.1 Elastoplastic coupling

The elastic properties of granular materials can be described by a hyperelastic nonlinear law providing a generalization to finite strains of the corresponding equation introduced in Part I, Section 2.5. This generalization is represented by the following potential

$$\begin{aligned}\phi(\boldsymbol{\epsilon}_e, \boldsymbol{\epsilon}_p) = &-\frac{\mu}{3}(\operatorname{tr}\boldsymbol{\epsilon}_e)^2 + c\operatorname{tr}\boldsymbol{\epsilon}_e \\ &+ (p_0 + c)\left[\left(d - \frac{1}{d}\right)\frac{(\operatorname{tr}\boldsymbol{\epsilon}_e)^2}{2\tilde{\kappa}} + d^{1/n}\tilde{\kappa}\exp\left(-\frac{\operatorname{tr}\boldsymbol{\epsilon}_e}{d^{1/n}\tilde{\kappa}}\right)\right] + \mu\,\boldsymbol{\epsilon}_e\cdot\boldsymbol{\epsilon}_e,\end{aligned} \tag{42}$$

where $\tilde{\kappa}$ is the elastic logarithmic bulk modulus, p_0 is the initial confining pressure, c, d, and μ are scalar parameters depending on the volumetric plastic strain $\operatorname{tr}\boldsymbol{\epsilon}_p = \operatorname{tr}\boldsymbol{E}_p^{(0)}$,

providing the elastoplastic coupling. The dependence of parameters c, d, and μ on the volumetric plastic strain is made explicit by the following equations

$$\begin{aligned} &\mu = \mu_0 + c\left(d - \frac{1}{d}\right)\mu_1, \quad d = 1 + B < p_c - p_{cb} >, \\ &c = c_\infty \left[1 - \exp\left(-\Gamma < p_c - p_{cb} >\right)\right], \end{aligned} \tag{43}$$

and

$$\exp\left(\operatorname{tr} \boldsymbol{E}_p^{(0)}\right) - 1 = -\tilde{a}_1 \exp\left(-\frac{\Lambda_1}{p_c}\right) - \tilde{a}_2 \exp\left(-\frac{\Lambda_2}{p_c}\right), \tag{44}$$

where μ_0, μ_1, B, p_{cb}, c_∞, Γ, $\tilde{a}_1$, $\tilde{a}_2$, Λ_1, and Λ_2 are positive material constants. It can be noted that eqns. (43) do not include 'geometrical terms' and thus coincide with those of the small strain formulation, whereas eqn. (44) has been consistently generalized.

The potential (42) represents a isotropic function of the logarithmic strain and can be written as the sum of a volumetric and a deviatoric component. The deviatoric potential coincides with that employed by Ortiz and Pandolfi (2004) in the special case of null cohesion $c = 0$ and null coupling $d = 1$.

The Kirchhoff stress can be obtained from the potential (42) as

$$\boldsymbol{K} = \frac{\partial \phi}{\partial \boldsymbol{\epsilon}_e}, \tag{45}$$

so that it results in the form

$$\boldsymbol{K} = \left\{-\frac{2}{3}\mu \operatorname{tr} \boldsymbol{\epsilon}_e + c + (p_0 + c)\left[\left(d - \frac{1}{d}\right)\frac{\operatorname{tr} \boldsymbol{\epsilon}_e}{\tilde{\kappa}} - \exp\left(-\frac{\operatorname{tr} \boldsymbol{\epsilon}_e}{d^{1/n}\tilde{\kappa}}\right)\right]\right\}\boldsymbol{I} + 2\mu\, \boldsymbol{\epsilon}_e. \tag{46}$$

Eqn. (46) implies that the Kirchhoff stress and the logarithmic strain are coaxial, so that these become in the present context work-conjugate stress and strain measures, eqn. (9).

The elastic constitutive law (46) can be written in the form (32) with

$$\begin{aligned} \hat{\boldsymbol{K}}(\boldsymbol{F}_e, k_i) = \Bigg\{ &-\frac{1}{3}\mu \operatorname{tr}\log \boldsymbol{F}_e \boldsymbol{F}_e^T + c \\ &+(p_0 + c)\left[\left(d - \frac{1}{d}\right)\frac{\operatorname{tr}\log \boldsymbol{F}_e \boldsymbol{F}_e^T}{2\tilde{\kappa}} - \exp\left(-\frac{\operatorname{tr}\log \boldsymbol{F}_e \boldsymbol{F}_e^T}{2d^{1/n}\tilde{\kappa}}\right)\right]\Bigg\}\boldsymbol{I} \\ &+\mu \log \boldsymbol{F}_e \boldsymbol{F}_e^T. \end{aligned} \tag{47}$$

where the set k_i is now including c, d, and μ. Obviously, eqn. (47) can be expressed in the form (41), not reported for conciseness.

We are now in a position to write down the fourth-order elastic tensor $\mathbb{E}$ defined by eqn. $(12)_1$. This takes the form

$$\begin{aligned} \mathbb{E} = &-\frac{1}{2}\left(\boldsymbol{U}^{-1} \,\underline{\overline{\otimes}}\, \hat{\boldsymbol{K}}\boldsymbol{U}^{-1} + \hat{\boldsymbol{K}}\boldsymbol{U}^{-1} \,\underline{\overline{\otimes}}\, \boldsymbol{U}^{-1}\right) \\ &+\frac{1}{2}\left(\frac{\partial \boldsymbol{U}^{-1}\hat{\boldsymbol{K}}(\boldsymbol{X}, k_i)}{\partial \boldsymbol{X}} + \frac{\partial \hat{\boldsymbol{K}}(\boldsymbol{X}, k_i)\boldsymbol{U}^{-1}}{\partial \boldsymbol{X}}\right)_{\boldsymbol{X} = \boldsymbol{U}\boldsymbol{U}_p^{-1}} \left(\boldsymbol{I} \,\underline{\overline{\otimes}}\, \boldsymbol{U}_p^{-1}\right), \end{aligned} \tag{48}$$

so that, employing eqn. (47), we obtain

$$\frac{\partial \hat{\boldsymbol{K}}(\boldsymbol{X}, k_i)}{\partial \boldsymbol{X}} = \left\{ \left[-\frac{\mu}{3} + K_t(\boldsymbol{X}) \right] \boldsymbol{I} \otimes \boldsymbol{I} + \mu \boldsymbol{I} \,\overline{\underline{\otimes}}\, \boldsymbol{I} \right\} \left(\frac{\partial \log \boldsymbol{Y}}{\partial \boldsymbol{Y}} \right)_{\boldsymbol{Y}=\boldsymbol{X}\boldsymbol{X}^T} (\boldsymbol{I} \,\underline{\otimes}\, \boldsymbol{X} + \boldsymbol{X} \,\overline{\otimes}\, \boldsymbol{I}) , \tag{49}$$

where

$$K_t(\boldsymbol{X}) = \frac{p_0 + c}{2\tilde{\kappa}} \left[d - \frac{1}{d} + d^{-1/n} \exp\left(-\frac{\operatorname{tr} \log \boldsymbol{X}\boldsymbol{X}^T}{2d^{1/n}\tilde{\kappa}} \right) \right] . \tag{50}$$

and

$$\frac{\partial \log \boldsymbol{Y}}{\partial \boldsymbol{Y}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{r=0}^{n-1} (\boldsymbol{Y} - \boldsymbol{I})^r \,\underline{\overline{\otimes}}\, (\boldsymbol{Y} - \boldsymbol{I})^{n-1-r} , \tag{51}$$

a formula that can be found in (Hoger, 1987) or deduced from (Itskov and Aksel, 2002). Note that four tensorial products between second-order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ have been employed, which can be defined, with reference to every tensor $\boldsymbol{C}$, as

$$\begin{aligned} &(\boldsymbol{A} \otimes \boldsymbol{B})[\boldsymbol{C}] = (\boldsymbol{C} \cdot \boldsymbol{B}^T)\boldsymbol{A}, \qquad && (\boldsymbol{A} \,\underline{\overline{\otimes}}\, \boldsymbol{B})[\boldsymbol{C}] = \frac{1}{2}\boldsymbol{A}\left(\boldsymbol{C} + \boldsymbol{C}^T\right)\boldsymbol{B}^T, \\ &(\boldsymbol{A} \,\underline{\otimes}\, \boldsymbol{B})[\boldsymbol{C}] = \boldsymbol{A}\boldsymbol{C}\boldsymbol{B}^T, && (\boldsymbol{A} \,\overline{\otimes}\, \boldsymbol{B})[\boldsymbol{C}] = \boldsymbol{A}\boldsymbol{C}^T\boldsymbol{B}^T, \end{aligned} \tag{52}$$

so that the following property holds

$$\underline{\overline{\otimes}} = \frac{1}{2} (\,\underline{\otimes} + \overline{\otimes}) . \tag{53}$$

5.2 The irreversible strain rate

Eqn. $(15)_1$ defines the irreversible strain rate, which, accordingly, may be calculated taking the derivatives of eqn. (41) in which $\hat{\boldsymbol{K}}$ is given by eqn. (47). We obtain

$$\dot{\Lambda}\boldsymbol{P} = \mathbb{G}\left[\dot{\boldsymbol{E}}_p^{(0)}\right] , \tag{54}$$

where tensor $\mathbb{G}$, assumed positive definite, is given by

$$\begin{aligned}
\mathbb{G} \;=\; & -\frac{1}{2}\xi_2\,\mathbb{E}^{-1}\left[\left(\boldsymbol{U}^{-1}\frac{\partial\hat{\boldsymbol{K}}}{\partial c}+\frac{\partial\hat{\boldsymbol{K}}}{\partial c}\boldsymbol{U}^{-1}\right)\otimes\boldsymbol{I}\right] \\
& -\frac{1}{2}\xi_3\,\mathbb{E}^{-1}\left[\left(\boldsymbol{U}^{-1}\frac{\partial\hat{\boldsymbol{K}}}{\partial d}+\frac{\partial\hat{\boldsymbol{K}}}{\partial d}\boldsymbol{U}^{-1}\right)\otimes\boldsymbol{I}\right] \\
& -\frac{1}{2}\xi_4\,\mathbb{E}^{-1}\left[\left(\boldsymbol{U}^{-1}\frac{\partial\hat{\boldsymbol{K}}}{\partial \mu}+\frac{\partial\hat{\boldsymbol{K}}}{\partial \mu}\boldsymbol{U}^{-1}\right)\otimes\boldsymbol{I}\right] \\
& +\frac{1}{2}\,\mathbb{E}^{-1}\left(\frac{\partial\boldsymbol{U}^{-1}\hat{\boldsymbol{K}}(\boldsymbol{X})}{\partial\boldsymbol{X}}+\frac{\partial\hat{\boldsymbol{K}}(\boldsymbol{X})\boldsymbol{U}^{-1}}{\partial\boldsymbol{X}}\right)_{\boldsymbol{X}=\boldsymbol{U}\boldsymbol{U}_p^{-1}}\cdot \\
& \cdot\left(\boldsymbol{U}\boldsymbol{U}_p^{-1}\,\underline{\overline{\otimes}}\,\boldsymbol{U}_p^{-1}\right)\frac{\partial\exp\boldsymbol{E}_p^{(0)}}{\partial\boldsymbol{E}_p^{(0)}},
\end{aligned} \tag{55}$$

in which

$$\frac{\partial\hat{\boldsymbol{K}}}{\partial c}= \left\{1+\left[\left(d-\frac{1}{d}\right)\frac{\operatorname{tr}\log\boldsymbol{U}\boldsymbol{U}_p^{-2}\boldsymbol{U}}{2\tilde{\kappa}}-\exp\left(-\frac{\operatorname{tr}\log\boldsymbol{U}\boldsymbol{U}_p^{-2}\boldsymbol{U}}{2d^{1/n}\tilde{\kappa}}\right)\right]\right\}\boldsymbol{I}, \tag{56}$$

$$\frac{\partial\hat{\boldsymbol{K}}}{\partial d}=\left\{\frac{(p_0+c)\operatorname{tr}\log\boldsymbol{U}\boldsymbol{U}_p^{-2}\boldsymbol{U}}{2\tilde{\kappa}}\left[1+\frac{1}{d^2}-\frac{1}{2nd^{1+1/n}}\exp\left(-\frac{\operatorname{tr}\log\boldsymbol{U}\boldsymbol{U}_p^{-2}\boldsymbol{U}}{2d^{1/n}\tilde{\kappa}}\right)\right]\right\}\boldsymbol{I}, \tag{57}$$

$$\frac{\partial\hat{\boldsymbol{K}}}{\partial\mu}=\left(-\frac{1}{3}\operatorname{tr}\log\boldsymbol{U}\boldsymbol{U}_p^{-2}\boldsymbol{U}\right)\boldsymbol{I}+\log\boldsymbol{U}\boldsymbol{U}_p^{-2}\boldsymbol{U}, \tag{58}$$

and

$$\xi_2=-\frac{c_\infty\Gamma\,H(p_c-p_{cb})\exp\left[-\Gamma(p_c-p_{cb})\right]p_c^2\exp(\operatorname{tr}\boldsymbol{E}_p^{(0)})}{\tilde{a}_1\Lambda_1\exp\left(-\dfrac{\Lambda_1}{p_c}\right)+\tilde{a}_2\Lambda_2\exp\left(-\dfrac{\Lambda_2}{p_c}\right)}, \tag{59}$$

$$\xi_3=-\frac{B\,H(p_c-p_{cb})\,p_c^2\exp(\operatorname{tr}\boldsymbol{E}_p^{(0)})}{\tilde{a}_1\Lambda_1\exp\left(-\dfrac{\Lambda_1}{p_c}\right)+\tilde{a}_2\Lambda_2\exp\left(-\dfrac{\Lambda_2}{p_c}\right)}, \tag{60}$$

$$\xi_4 = \left(d - \frac{1}{d}\right)\mu_1\xi_2 + c\left(1 + \frac{1}{d^2}\right)\mu_1\xi_3. \tag{61}$$

Note that the exponential of a tensor [see e.g. (Itskov and Aksel, 2002) for the definition] has been introduced in eqn. (55), together with its gradient, defined as

$$\frac{\partial \exp \boldsymbol{E}_p^{(0)}}{\partial \boldsymbol{E}_p^{(0)}} = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{r=0}^{n-1} \left(\boldsymbol{E}_p^{(0)}\right)^r \,\overline{\otimes}\, \left(\boldsymbol{E}_p^{(0)}\right)^{n-1-r}, \tag{62}$$

a formula given by Itskov and Aksel (2002).

5.3 The yield function

In the absence of *ad hoc* experimental results, we employ for simplicity a yield function with same form adopted for the infinitesimal theory (see Part I of this paper and Bigoni and Piccolroaz, 2004), where now the Cauchy stress is replaced by the Biot stress $\boldsymbol{T}^{(1)}$. This can be pursued re–defining the invariants p, q and θ in terms of Biot stress

$$p = -\frac{\operatorname{tr} \boldsymbol{T}^{(1)}}{3}, \quad q = \sqrt{3J_2}, \quad \theta = \frac{1}{3}\cos^{-1}\left(\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right), \tag{63}$$

where $\theta \in [0, \pi/3]$ and

$$\begin{aligned} &J_2 = \frac{1}{2}\operatorname{dev} \boldsymbol{T}^{(1)} \cdot \operatorname{dev} \boldsymbol{T}^{(1)}, \quad J_3 = \frac{1}{3}\operatorname{tr}\left(\operatorname{dev} \boldsymbol{T}^{(1)}\right)^3, \\ &\operatorname{dev} \boldsymbol{T}^{(1)} = \boldsymbol{T}^{(1)} - \frac{\operatorname{tr} \boldsymbol{T}^{(1)}}{3}\boldsymbol{I}. \end{aligned} \tag{64}$$

As a consequence, the yield function takes a form of the type $f_{\boldsymbol{T}^{(1)}}(\boldsymbol{T}^{(1)}, \mathcal{K}) \leq 0$, namely,

$$F(\boldsymbol{T}^{(1)}, p_c, c) = f(p, p_c, c) + \frac{q}{g(\theta)}, \tag{65}$$

where p_c and c are the parameters governing the change in shape of the yield surface caused by the hardening (as in the infinitesimal theory), and

$$f(p, p_c, c) = \begin{cases} -Mp_c\sqrt{(\Phi - \Phi^m)\left[2(1-\alpha)\Phi + \alpha\right]} & \text{if } \Phi \in [0,1], \\ +\infty & \text{if } \Phi \notin [0,1], \end{cases} \tag{66}$$

in which

$$\Phi = \frac{p + c(\operatorname{tr} \boldsymbol{E}_p^{(0)})}{p_c(\operatorname{tr} \boldsymbol{E}_p^{(0)}) + c(\operatorname{tr} \boldsymbol{E}_p^{(0)})} \tag{67}$$

and

$$g(\theta) = \frac{1}{\cos\left[\beta\frac{\pi}{6} - \frac{1}{3}\cos^{-1}\left(\gamma\cos 3\theta\right)\right]}. \tag{68}$$

Note that M, m, α, β, and γ are material parameters with the same meaning as in the infinitesimal theory (already described in Part. I of this paper and by Bigoni and Piccolroaz, 2004).

The yield function gradient

$$\boldsymbol{Q} = \frac{\partial F(\boldsymbol{T}^{(1)}, p_c, c)}{\partial \boldsymbol{T}^{(1)}}, \tag{69}$$

can be obtained directly from the yield function (66) or from the corresponding equations in Part I of this paper (Appendix A), with $\boldsymbol{T}^{(1)}$ replacing $\boldsymbol{\sigma}$.

5.4 Flow rule and hardening modulus

The flow mode tensor $\boldsymbol{P}$ is postulated in the form

$$\boldsymbol{P} = \boldsymbol{Q} - \frac{\operatorname{tr}\boldsymbol{Q}}{3}\epsilon\left(1-\Phi\right)\boldsymbol{I}, \tag{70}$$

where $0 \leq \epsilon \leq 1$ is a nonassociativity parameter, null for associative flow rule.

The hardening modulus can be calculated from the definition (21) in the form

$$\dot{\Lambda} h = -\left(\frac{\partial F}{\partial p_c}\dot{p}_c + \frac{\partial F}{\partial c}\dot{c}\right), \tag{71}$$

where

$$\begin{aligned}\frac{\partial F}{\partial p_c} &= -M\sqrt{\left(\Phi-\Phi^m\right)\left[2(1-\alpha)\Phi+\alpha\right]} \\ &+M\frac{p_c(p+c)}{(p_c+c)^2}\frac{\left(1-m\Phi^{m-1}\right)\left[2(1-\alpha)\Phi+\alpha\right]+2(1-\alpha)\left(\Phi-\Phi^m\right)}{2\sqrt{\left(\Phi-\Phi^m\right)\left[2(1-\alpha)\Phi+\alpha\right]}},\end{aligned} \tag{72}$$

and

$$\begin{aligned}\frac{\partial F}{\partial c} &= -M\frac{p_c(p_c-p)}{(p_c+c)^2}\cdot \\ &\cdot\frac{\left(1-m\Phi^{m-1}\right)\left[2(1-\alpha)\Phi+\alpha\right]+2(1-\alpha)\left(\Phi-\Phi^m\right)}{2\sqrt{\left(\Phi-\Phi^m\right)\left[2(1-\alpha)\Phi+\alpha\right]}},\end{aligned} \tag{73}$$

in which

$$\dot{p}_c = -\frac{p_c^2\exp(\operatorname{tr}\boldsymbol{E}_p^{(0)})}{\tilde{a}_1\Lambda_1\exp\left(-\dfrac{\Lambda_1}{p_c}\right)+\tilde{a}_2\Lambda_2\exp\left(-\dfrac{\Lambda_2}{p_c}\right)}\operatorname{tr}\dot{\boldsymbol{E}}_p^{(0)}, \tag{74}$$

and

$$\dot{c} = c_\infty \Gamma \, H(p_c - p_{cb}) \exp\left[-\Gamma(p_c - p_{cb})\right] \dot{p}_c. \tag{75}$$

Parameters Λ_1, Λ_2, $\tilde{a}_1$, $\tilde{a}_2$, c_∞, Γ, and p_{cb} have been introduced and motivated in Part I of this paper.

5.5 The rate constitutive equations for the compaction model

The elastoplastic incremental constitutive equations (18), written in terms of Biot stress $\boldsymbol{T}^{(1)}$ and conjugate strain $\boldsymbol{E}^{(1)} = \boldsymbol{U} - \boldsymbol{I}$, take the form

$$\dot{\boldsymbol{T}}^{(1)} = \begin{cases} \mathbb{E}[\dot{\boldsymbol{E}}^{(1)}] - \dfrac{1}{g} < \boldsymbol{Q} \cdot \mathbb{E}[\dot{\boldsymbol{E}}^{(1)}] > \mathbb{E}[\boldsymbol{P}] & \text{if } F(\boldsymbol{T}^{(1)}, p_c, c) = 0, \\ \mathbb{E}[\dot{\boldsymbol{E}}^{(1)}] & \text{if } F(\boldsymbol{T}^{(1)}, p_c, c) < 0, \end{cases} \tag{76}$$

where the elastic tensor $\mathbb{E}$ is given by eqns. (48)–(51), the yield function $F(\boldsymbol{T}^{(1)}, p_c, c)$ by eqns. (65)–(68) and the yield function gradient $\boldsymbol{Q}$ and flow mode tensor $\boldsymbol{P}$ by eqns. (69) and (70). The plastic modulus g is provided by eqn. (20), where h, the hardening modulus, is obtained substituting eqn. (54) into eqns. (74)–(75) and eqn. (71), thus yielding

$$h = -\left(\frac{\partial F}{\partial p_c}\bar{p}_c + \frac{\partial F}{\partial c}\bar{c}\right), \tag{77}$$

where $\partial F/\partial p_c$ and $\partial F/\partial c$ are specified by eqns. (72)–(73) and

$$\begin{aligned} \bar{p}_c &= -\frac{p_c^2 \exp(\operatorname{tr} \boldsymbol{E}_p^{(0)})}{\tilde{a}_1 \Lambda_1 \exp\left(-\dfrac{\Lambda_1}{p_c}\right) + \tilde{a}_2 \Lambda_2 \exp\left(-\dfrac{\Lambda_2}{p_c}\right)} \operatorname{tr} \mathbb{G}^{-1}[\boldsymbol{P}], \\ \bar{c} &= c_\infty \Gamma \, H(p_c - p_{cb}) \exp\left[-\Gamma(p_c - p_{cb})\right] \bar{p}_c, \end{aligned} \tag{78}$$

in which tensor $\mathbb{G}$ is explicited by eqns. (55)–(62).

6 Conclusions

A new, consistent generalization to large strains of elastoplasticity theory with coupling between elastic and plastic properties has been given, based on work-conjugate variables and isotropy of the elastic response. This has permitted the extension to large strains of the model introduced in Part I of this paper, to describe granular materials becoming cohesive during mechanical, cold densification. Therefore, the model allows the simulation of forming processes of green bodies from ceramic powders, including situations in which large strains are involved.

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